featured articles

HOW PRACTICAL ARE MINIMUM-ERROR MAP PROJECTIONS?

John P. Snyder

John P. Snyder is retired from the National Mapping Division, U.S. Geological Survey, Reston, VA. 17212 Friends House Road, Sandy Spring, MD 20860

Ever since the Mercator projection gained wide acceptance for general geographic world maps, there have been attempts to replace it because of its serious area distortion. Most minimum-error projections, however, are difficult or nearly impossible to construct without a modern computer. Does this negate their use? The answer is probably yes if most users need to digitize maps or do their own programming of formulas, but no if the goal is to make the map easier for measurement of distance, area, and shape. We too often still choose projections to suit pre-computer criteria involving ease of construction, rather than to meet the needs of the map user. This paper reviews the practicality of minimum-error map projections and illustrates a wide range of minimum-error projections.

WORLD MAPS

Making a flat map resemble the round world has been a goal through the years for map makers. The smaller the region being portrayed, the more the flat map can look like that part of the globe. When the portion is decreased to a province or town, the distortion is often so small, although cartographically significant, that it can be perceived only by measurement, not by appearance. The differences among map projections are most evident when comparing various world maps.

The classic world map is of course based on the Mercator projection (Figure 1), presented in 1569 by the Flemish map maker Gerardus Mercator as a navigational aid, not as a general world map. With the importance of navigation, especially during the 15th and 16th centuries, the Mercator projection gained such high visibility that it became the standard for maps of world geography and has never really lost that role. Cartographers have regularly decried its general use, writing in prominent technical books of the past century, but it is so entrenched that Arno Peters had fertile ground for attacking the Mercator's gross area distortion as a basis for independently re-presenting Gall's hundred-year-old Orthographic Cylindrical projection (Figure 2), with the implication that Peters' approach was the first equal-area solution to supplant the Mercator.

Peters' presentation, beginning in 1973, was only the most vocal of several attempts to counteract the area distortion of the Mercator projection. Several innovators explicitly stated that their world map projections were attempts to resemble the Mercator with less distortion: Gall's Stereographic projection of 1855 (Figure 3), Van der Grinten's circular projection of 1898 (Figure 4), and O.M. Miller's "modified Mercator" cylindrical of 1942 (Figure 5) are familiar examples. These compromise projections made no claim to minimum error; they tried to reduce the visual distortion. Further steps were taken by numerous inventors who retained straight, parallel lines of latitude, but who curved the meridians, producing what are generally called pseudocylindrical projections. The area scale is frequently true throughout the world map, but angles and shapes are often badly distorted. The Sinusoidal projection (Figure 6), the Mollweide projection (Figure 7), and Eckert's Nos. 4 and 6 (Figures 8 and

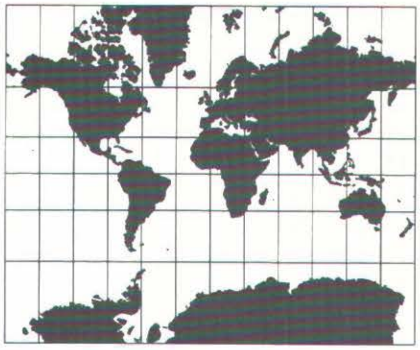


Figure 1. Mercator projection.

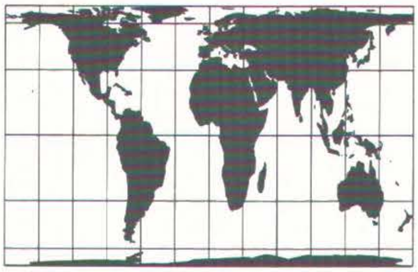


Figure 2. Gall Orthographic (Cylindrical Equal-Area) projection.

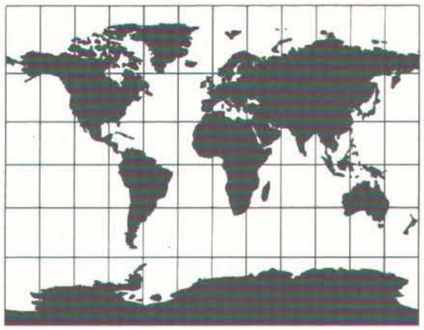


Figure 3. Gall Stereographic projection.

Figure 4. Van der Grinten projection.

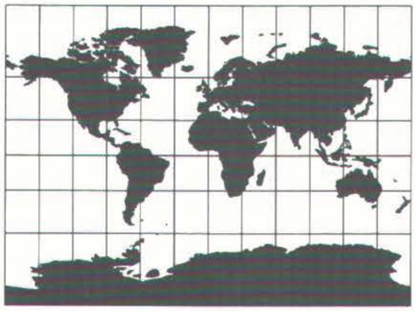


Figure 5. Miller Cylindrical projection.

Figure 6. Sinusoidal projection.

Figure 7. Mollweide projection.

Figure 8. Eckert IV projection.

Figure 9. Eckert VI projection.

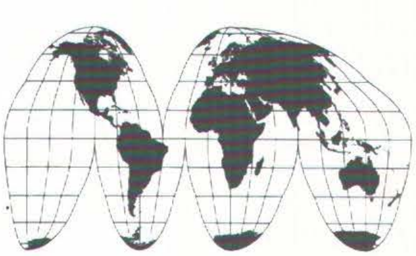


Figure 10. Goode Homolosine projection.

Figure 11. Robinson projection.

Figure 12. Eisenlohr projection.

9) are all equal-area. Goode's Homolosine (Figure 10) is also equal-area, but using interruptions he reduced the shape distortion. Interruption has its drawbacks, and Arthur H. Robinson designed what is now probably the best-known compromise pseudocylindrical in 1963 (Figure 11) for Rand McNally; it is neither interrupted nor equal-area. Rand McNally used the projection on a limited basis, but when the National Geographic Society adopted it in 1988 with an effective press conference, it became far better known.

There have been minimum-error world map projections, however, beginning a few decades after the development of minimum error as a mathematical concept in the early 19th century.[1] When applied to map projections, innovators soon found that the concept had to be applied narrowly. In 1870 Eisenlohr presented a minimum-error conformal world map projection by figuring out how to have the scale constant all around the edge of the map. The only problem is that the map (Figure 12) looks awful. A conformal map projection is one on which all small shapes and local angles are shown correctly. The Mercator is one example. In 1910 Behrmann presented a cylindrical equal-area projection with what he determined to be as little distortion of angles as possible, but it (Figure 13) is only a slight improvement over the Gall-Peters projection. With computers available to do the mathematics, there have several attempts since 1980 to develop minimum-error world maps which are neither conformal nor equal-area by Peters' son Aribert in Germany, by Canters in Belgium, by Laskowski in the U.S., and by others. So far they remain academic, lacking commercial application.

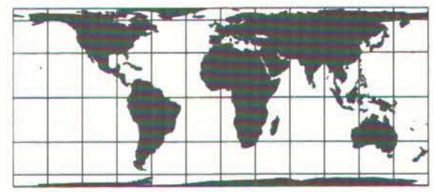


Figure 13. Behrmann Cylindrical Equal-Area projection.

REGIONAL MAPS

There is much more justification, however, for minimum-error regional map projections. The Russian mathematician Chebyshev had theorized in 1856 that a conformally mapped region bounded by a line of constant scale has the least overall error, or is minimum-error. If the region is circular, this is achieved with an azimuthal projection, because the projection of the globe onto a plane tangent at the pole (Figure 14) or somewhere else (Figure 15) produces an azimuthal projection on which lines of constant distortion are circles centered on the point of tangency (Figure 16). Chebyshev's theory was later proven, and it was applied in effect by Eisenlohr to his world map projection and by others in the 20th century to several projections used for map regions bounded by a rectangle or consisting of a particular landmass. In 1926 Laborde applied it to topographic mapping of the island of Madagascar, in 1953 O.M. Miller used the principle for an Oblated Stereographic projection of the combined continents of

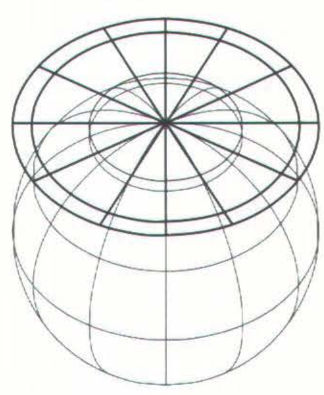


Figure 14. Concept for a polar azimuthal projection.

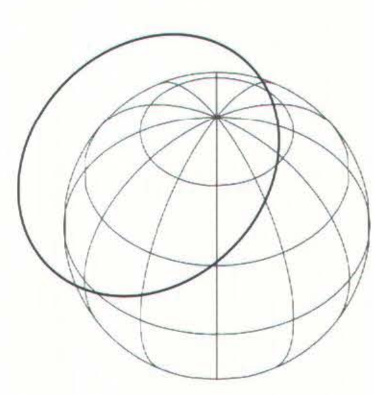


Figure 15. Concept for an oblique azimuthal projection.

[1] The concept of minimum error is closely tied to that of least squares, developed by mathematicians Gauss and Legendre early in the 19th century. This principle states that the best value for a quantity, given a set of measurements of that quantity, is the value for which the sum of the squares of deviations of these measurements from this value is least. For a minimum-error map projection, the sum of the squares of the deviations of all the actual scale values from the stated scale is made a minimum according to a prescribed definition.

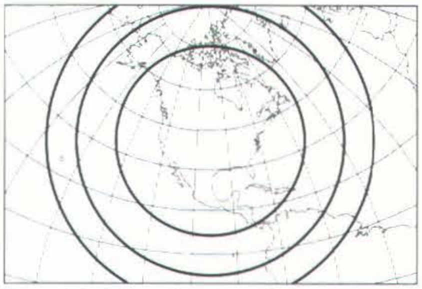


Figure 16. Oblique Stereographic projection, with lines of constant scale factor (1.1, 1.2, 1.3).

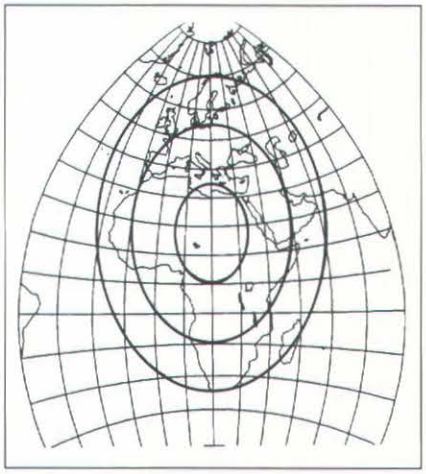


Figure 17. Miller Oblated Stereographic projection for Europe and Africa, with lines of constant scale factor (0.94, 1.00, 1.08).

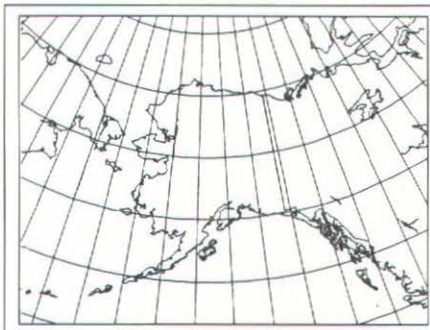
Europe and Africa (Figure 17), and it was later used, as computers facilitated the handling of more complicated equations, by Reilly for New Zealand, and in my development of low-error map projections for Alaska (Figure 18) and for the "lower-48" States (Figure 19).

The most recent application was published in 1992 as the Optimal Conformal projection developed by physicist Mitchell Feigenbaum for Hammond Inc. and used for the continental maps in the new *Hammond Atlas of the World*. In the Hammond maps, the bounding line roughly follows the continental coastlines, including related islands, and the maps were reasonably touted as "the most distortion-free that can ever be made," although the words "conformal maps" should really be inserted, and calculations are extremely complicated.

The Chebyshev principle appears to be applicable to equal-area map projections, although I haven't heard of an analytical proof, and I have used it to develop an Oblated Equal-Area projection for oval (Figure 20) and rectangular (Figure 21) regions. John Dyer developed formulas to apply the minimum-error concept to irregular regions, and I applied his system to Alaska (Figure 22).

As with world maps, arbitrary or compromise projections, neither equal-area nor conformal, can also be developed for regions on a minimum-error basis. George B. Airy in 1861 was the first with a minimum-error azimuthal projection (Figure 23) which looks very much like an Azimuthal Equidistant projection. A century later, Bomford of England and later Ginzburg of the Soviet Union (Figure 24) devised low-error compromise versions to suit rectangular or oval regions. Tobler devised an Optimal projection of the 48 States, minimizing scale variation between all the intersections of a 5° graticule of meridians and parallels.

If we are talking about a map of a full hemisphere, with a meridian, the equator, or an oblique great circle as its circular boundary, the solution is straightforward. The projection will be azimuthal, as discussed previously, and the minimum-error equal-area projection is the Lambert Azimuthal Equal-Area (Figure 25), the minimum-error conformal projection is the Stereographic (Figure 26), and the minimum-error projection in general is that by Airy.



Modified-Stereographic Conformal projection for Alaska, without lines of constant scale factor.

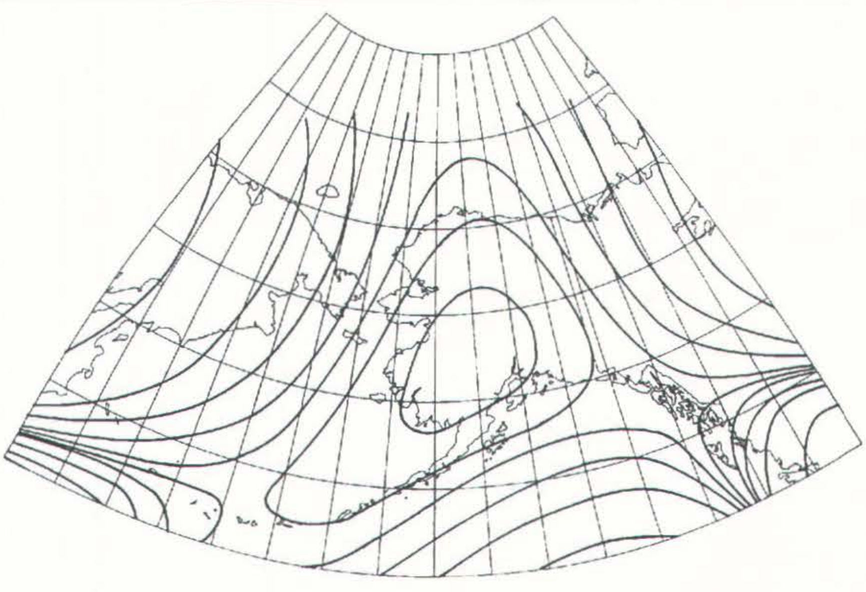


Figure 18. Modified-Stereographic Conformal projection for Alaska, with lines of constant scale factor.

The improvement obtained by using minimum-error projections varies considerably with the circumstances. For world maps, the use of the "minimum-error" Eisenlohr projection with its two large cusps may be rejected even over the Mercator almost out of hand, because of appearance. Minimum-error maps of hemispheres, just discussed, are standard and quite satisfactory in appearance. For regional maps, appearance is much less of a factor because even a large continent looks much the same on any of several good projections. When the region is the size of the United States or smaller, the choice among good projections makes almost no visual difference, but the range of scale can vary significantly. This variation can be quantified in different ways. Two useful measures of this variation are (1) the maximum and minimum values of the scale factor over a map, and (2) the root-mean-square error or RMSE of the scale factor. Although calculating the RMSE gives the computer a little mathematical workout, the concepts involved are not very complicated.

The scale factor on a map is the ratio of the actual scale at a given place to the nominal scale of the map. That is, if the nominal scale of the map is 1/250,000 or about 3.95 miles to the inch, and the scale at a particular place on the map is 3.85 miles to the inch, then the scale factor at that point is 0.976. The scale "error," so-called, although it isn't an error as much as just the way flat maps work, is then (1-.976) or 2.4%. If we take all these scale errors for small equal portions of the map, square them, add up the squares, divide by the number of measurements, and find the square root of the quotient, we get the root-mean-square error for the scale factors on the map. The mathematician Gauss found in the 1820s that this is an excellent measure of the overall error of a map or of many other sets of measurements, and that the lower, the better. Depending on the distribution of scale, the range between maximum and minimum values may not be least when the RMSE is least.

Applying these concepts to a specific case, we can first divide the land, islands, and adjacent waters of North America into about 240 quadrangles 5° of latitude x 5° of longitude in size, giving each a weight in proportion to its area on the Earth. Several world atlases use the Lambert Azimuthal Equal-Area projection for maps of this continent. If this projection is used, with the best possible center for these 240 quadrangles, the result is 4.5% for the RMSE or mean scale factor error, and a range of scale factors from 0.92 to 1.09, or 17%. With an Oblated Equal-Area projection, the RMSE drops from 4.5% to 3.3%, and the range is 14% instead of 17%. In other words, the true scale of the map is generally about 3.3% from the nominal scale on the Oblated Equal-Area, and about 4.5% on the Lambert Azimuthal Equal-Area. The resulting general improvement in scale error is a moderately significant 27%, although the extremes are only 18% closer. Because North America is more elongated than most other continents, the improvement using the Oblated Equal-Area projection rather than the Lambert is much more pronounced for North America. Therefore, in a recent selection for use in equal-area continental land-use maps by the U.S. Geological Survey's EROS Data Center at Sioux Falls, the Oblated Equal-Area was recommended for North America, and the Lambert was recommended for the rest of the continents.

If the region is reduced in size to the 48 conterminous United States, using almost a thousand 1° x 1° quadrangles of lat/long for the calculations, a suitable Oblated Equal-Area projection gives an RMSE of 0.81% in scale factor error and a range of 2.8%, while the commonly used Albers Equal-Area Conic has a mean error of 1.02% and a range of 3.1%, an improvement using the Oblated Equal-Area of 21% in mean error, and 10% in range. In this case, the improvement is moderately good, but the

ANALYSIS

Figure 19. Modified-Stereographic Conformal projection for 48 United States, with lines of constant scale factor (.99, 1.00, 1.011).

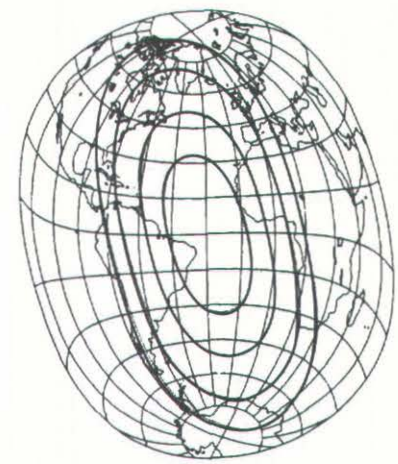


Figure 20. Oblated Equal-Area projection for Atlantic Ocean with lines of constant max. scale factor (1.02, 1.05, 1.10, 1.15).

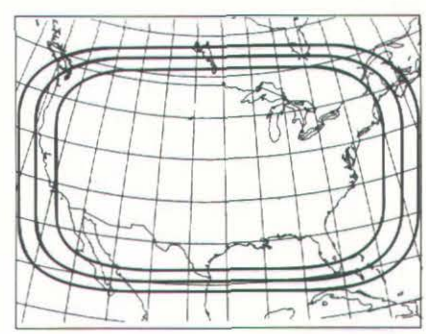


Figure 21. Oblated Equal-Area projection for 48 United States, with lines of constant max. scale factor (1.01, 1.0125, 1.015).

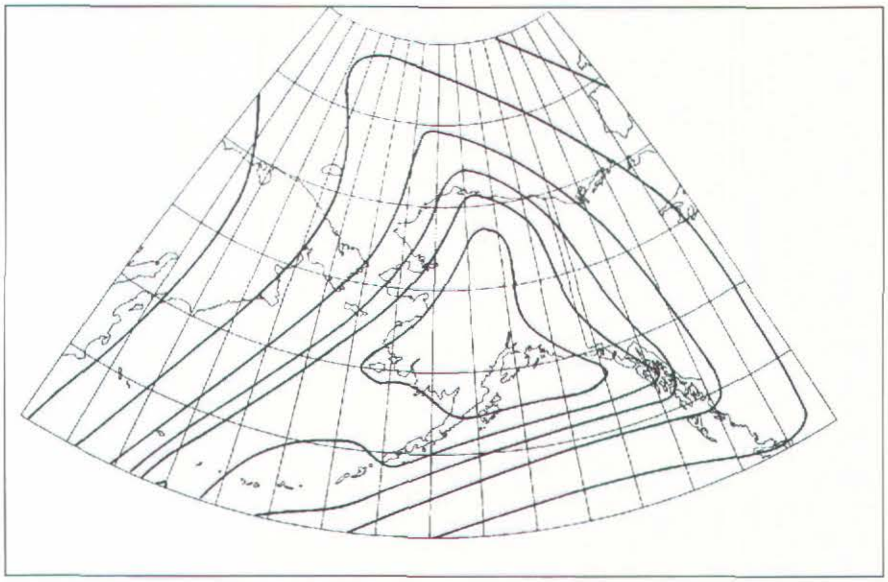


Figure 22. Low-Error Equal-Area projection for Alaska, with lines of constant max. scale factor.

scale is within 3% of the nominal scale anyway at any given point using either projection. To recommend a change of projection for the 48 States is hardly worthwhile.

With a region reduced to say a 10° x 10° quadrangle centered at 40°N. latitude (which is a region 30% longer in a north-south direction than east-west because of the narrower degrees of longitude) a Lambert Azimuthal Equal-Area projection, ideal for circular regions, is 60% better in mean error than an Albers, which is better for east-west regions, and the range is about 30% better. But, we are only talking about a scale factor ranging less than .5% within the entire quadrangle. A 1° x 1° quadrangle shows about the same RMSE improvement, but the scale factor range is 100 times closer to 1.

CONCLUSIONS

In conclusion, the choice of a map projection should be based on several criteria: the purpose of the map, the shape and size of the region being mapped, and whether the particular map is part of an established series or is to stand alone. For a new standalone map of a region, a minimum-error projection is clearly mathematically "better" than a projection that is not minimum-error. If all users are going to rely solely on the nominal map scale for measurement and will not be digitizing, and if the map maker can use the formulas for a minimum-error projection or software containing them to construct the map, then such a projection can be recommended for a region with a size of the order of North America. If these criteria are not met, the improvement in accuracy is probably offset by the mathematical complications, both in plotting, scale determination at a given point, or digitizing. In spite of the pervasiveness of computers, we still need to understand

Figure 23. Airy Minimum-Error Azimuthal projection for hemisphere centered on Washington, D.C.

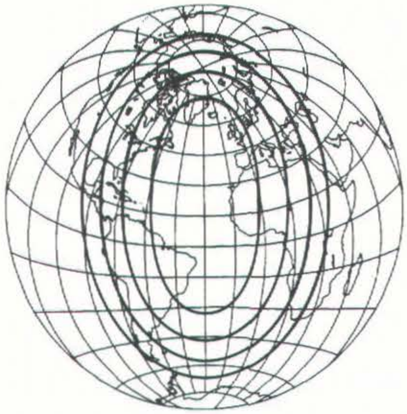


Figure 24. Ginzburg Pseudoazimuthal projection for Atlantic Ocean, with oval lines of constant maximum angular distortion (5°, 10°, 15°, 20°).

the map projection we are using. Projections are considered confusing enough by many cartographers because of the amount of math involved. A map used for accurate measurements must have a known projection, and it is necessary for the user to become familiar with the projection used. With all the tools, especially computers, available to us, we should not limit ourselves to pre-computer criteria in choosing the projection to be used, but we should know the pros and cons involved in the choice of a more complicated projection.

Figure 25. Oblique Lambert Azimuthal Equal-Area projection for hemisphere centered on Washington, D.C.

Figure 26. Oblique Stereographic projection for hemisphere centered on Washington, D.C.

REFERENCES

For further details and references see:

Snyder, J. 1987. *Map Projections - A Working Manual*. Washington: U.S. Geological Survey Professional Paper 1395.

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RESUMEN

El proyector Mercator ha ganado amplia aceptación para la proyección de mapas geográficos mundiales, pero han habido intentos de reemplazarlo debido a la seria distorsión del área. Sin embargo, la mayoría de errores de proyección, son muy difíciles o casi imposibles de detectar sin un computador moderno. Niega esto su uso? La respuesta probablemente es sí, si la mayoría de usuarios necesitan digitalizar mapas o hacer su propia programación de fórmulas, pero no, si la meta es hacer el mapa más fácil en medidas de distancia, área y forma. Nosotros todavía con frecuencia escogemos proyecciones que se ajustan al criterio pre-computarizado que ofrece facilidad en la construcción, a cambio de suplir las necesidades del cartógrafo. Este trabajo repasa la practicalidad de proyecciones de mapas con errores mínimos e ilustra una amplia variedad de ejemplos de proyección de errores mínimos.

RESUME

Depuis que la projection Mercator a reçu un accueil favorable du monde de la cartographie générale, des tentatives ont été faites dans le but de la remplacer à cause de la sévère déformation régionale qu'elle entraîne. La plupart des projections à erreur minimum, cependant, sont difficiles, même presqu'impossibles à construire sans l'aide d'un ordinateur moderne. Est-ce que cela nullifie leur utilité ? La réponse est probablement affirmative si la plupart des utilisateurs ont à convertir les cartes en numérique ou à programmer eux-mêmes leurs formules; elle est négative si le but est de faciliter sur la carte la mesure de la distance, de la région et de la forme. Trop souvent, nous continuons à choisir des projections qui respectent les critères antérieurs à l'ère de l'informatique qui impliquent la facilité de construction, au lieu de répondre aux besoins de l'utilisateur de la carte. L'article passe en revue les aspects pratiques des projections de cartes à erreur minimum et illustre une large gamme d'exemples de projections de ce type.