# Unusual Map Projections[1]

***Waldo Tobler***
*Professor Emeritus*
*University of California*
*Santa Barbara*
*www.geog.ucsb.edu/~tobler*

**In this paper I look at azimuthal projections, equal area projections, and spherical and map distances, with emphasis on less known variants.**

**Keywords**: Azimuthal projections, equal area maps, distance, simplifying coordinates.

## Introduction

To begin it is useful to remark on some basic facts. The surface of the earth is two dimensional, which is why only (but also both) latitude and longitude are needed to pin down a location. Many authors, and textbooks mistakenly refer to it as three-dimensional. Yes, it is embedded in three dimensions, but the surface is a curved, closed, and bumpy two-dimensional surface. The graticule on the earth rides up and down over hill and dale. Map projections convert this surface to a flat two-dimensional surface. All map projections preserve the two dimensionally of the surface.

All map projections also result in distorted maps. Since the time of Ptolemy the objective has been to obtain maps with as little distortion as possible. But Mercator changed this by introducing the idea of a systematic distortion to assist in the solution of a problem. Mercator's famous anamorphose is a nomogram that helps solve a navigation problem. His idea caught on. Thus it is useful to think of a map projection as you are used to thinking of graph paper: logarithmic and semi-logarithmic scales and probability plots and so on, are employed to bring out different aspects of data being analyzed. Map projections can be used in a similar manner to solve problems and are not only for geographic display. This, however, is not a common use in Geographic Information Systems.

*"But Mercator changed this by introducing the idea of a systematic distortion to assist in the solution of a problem."*

## Unusual Projections

Azimuthal map projections always show correct directions from their center. What varies is the map distance, relative to the spherical distance. The most common form represents the map within a circle. Thus the cylindrical-like azimuthal projection developed by J. Craig (1910) in Cairo, shown here with the center at the intersection of the Greenwich meridian and the Equator is unusual (Figure 1). A different center using Craig's projection will yield a different shape but will remain an azimuthal projection.

The radial distance on the different 'circular' azimuthal projections is extremely variable. Over two dozen have been named. In textbooks the conventional representation is to show the gnomonic, stereographic, equi-

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[1]Based on an invited presentation at the 1999 meeting of the Association of American Geographers in Hawaii. The full presentation can be seen at http://www.geog.ucsb.edu/~tobler/presentations/ and titled Unusual Map Projections, Honolulu, 1999.

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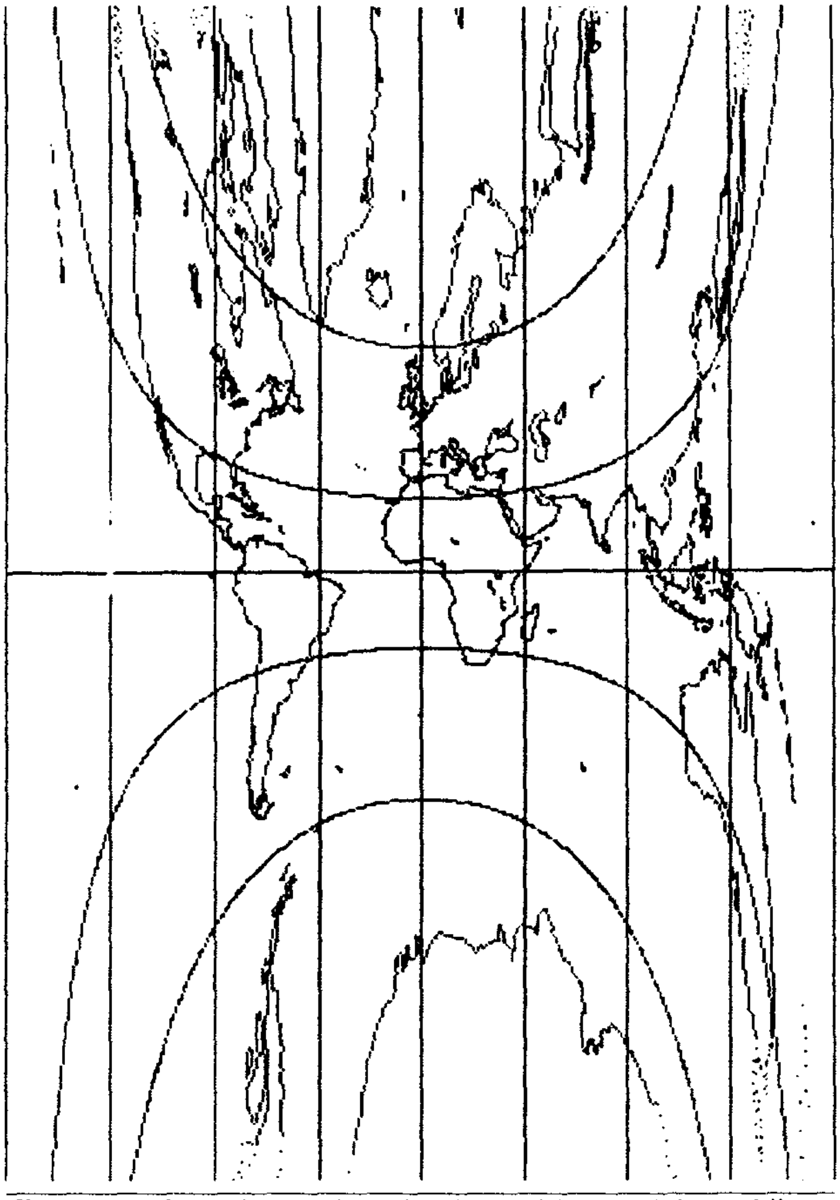

*Figure 1. An unusual azimuthal projection invented by J. Craig (1910). Azimuths from the center are correctly depicted.*

distant, orthographic and Lambert equal area projections by a polar view of latitudinal circles. On this type of diagram the variation in the distances from the center of the map is shown by variation in the spacing of the parallels. An alternate view is showing the curves in a graph of map distances versus spherical distances (Figure 2).

The X-axis represents the distance on the sphere, and the Y-axis represents the same distance (to scale) on the map. Take an increment (one centimeter, say) on the X-axis, and then move up to the curve. Then move across to the Y-axis to find the amount by which the spherical distance has changed. The advantage of this representation is that the slope of the curve quickly reveals the distance change. It is also an approximation to the areal enlargement. For example, if the slope is greater than one, the map area is enlarged. If the slope is less than one the map distances shrink. If the slope is equal to one we have the azimuthal equidistant pro-

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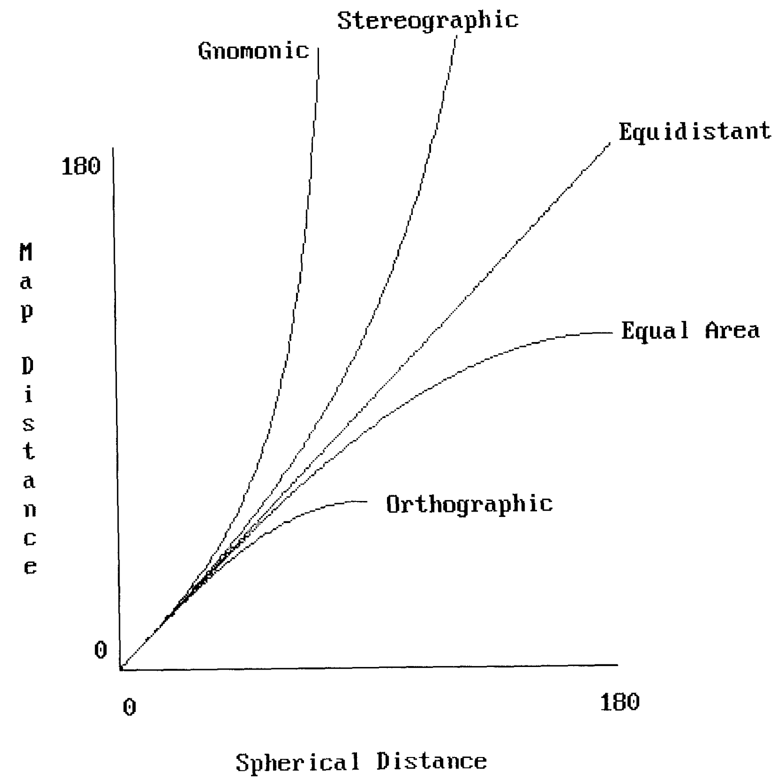


*Figure 2. Radial distance display of azimuthal projections. Spherical distances along the X axis, map distances along the Y axis.*

jection. In this view, Snyder's (1987) 'Magnifying Glass' projection appears as a kinked line (Figure 3).

In studying migration about the Swedish city of Asby, Hägerstrand (1957) used the logarithm of the actual distance as the radial scale. This enlarges the scale in the center of Asby, near which most of the migration takes place. Actually, but not generally shown, there is a small hole in the middle of the map since the logarithm of zero is minus infinity. This logarithmic azimuthal projection can easily be represented in the same graphic form as Snyder's 'Magnifying Glass' projection.

*"Retro-azimuthal projections show the direction to, not from, a center."*

Figure 4 shows two new map versions in the same form as quarter circles, one giving an azimuthal myopic view $\{r=(2\pi-\rho^2)^{1/2}\}$ and the other an anti-myopic view $\{r=\pi-(\pi^2-\rho^2)^{1/2}\}$. Popular today are also azimuthal maps on which the distance from the center is represented as fractional powers such the square or cube root of the spherical distance (Figure 5). It is also possible to scale azimuthal maps in terms of cost distances.

Retro-azimuthal projections show the direction to, not from, a center. For these maps it is also possible to choose different the distances to the center. One use was to let British colonials know in which direction to point their radio antennas to receive a signal sent from Rugby in the U.K (Hinks, 1929; Reeves, 1929). These unusual projections generally contain a hole inside of the map and a portion of the area overlaps itself (Tobler, 2002). The size of the overlap, and the void, depends on the latitude of the map center. Several retro-azimuthal projections are demonstrated in a computer program from Axion Spatial Imaging.

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Equal area projections are such that map areas are proportional to spherical areas. They are obtained by setting the differential of surface area on a sphere equal to that of a flat map. The consequent differential equa-

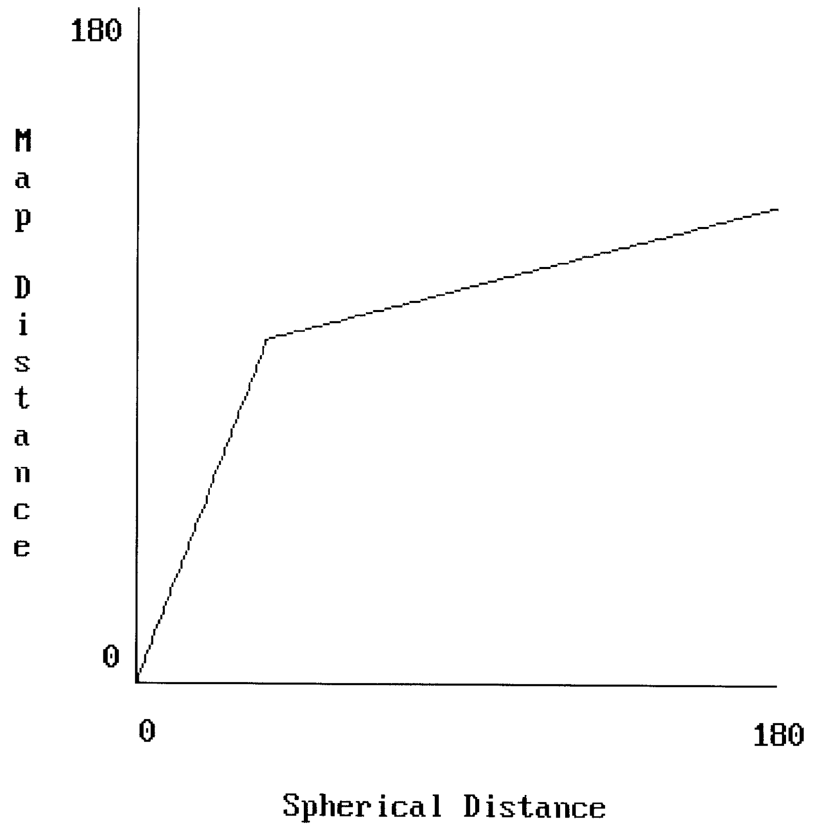


*Figure 3. Snyder's magnifying glass azimuthal projection in the radial distance form, with two scales and a discontinuity.*

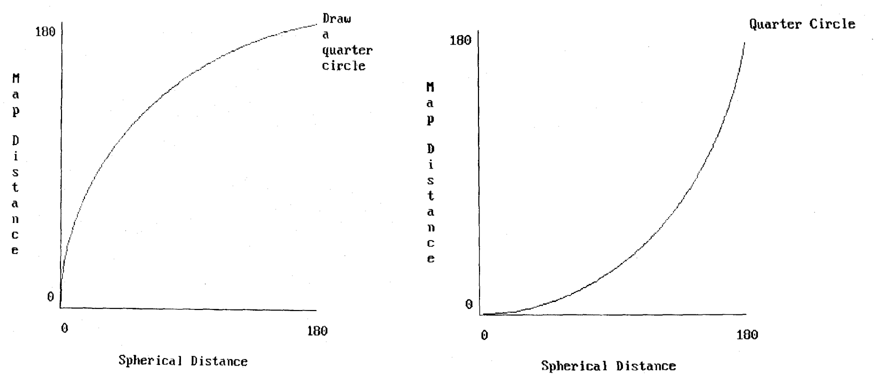


*Figure 4. Two new azimuthal projections: myopia version (left) and anti-myopia version (right).*

tion has many solutions and thus depends on additional conditions. One such condition is to fit the maps into a particular shape. Quite a number of such shapes have been obtained. Here are a few new ones. It is relatively easy to fit equal area maps into regular N sided polygons. One computer program can do them all, starting with a triangle, for which N = 3. The case of a pentagon (N = 5) is shown here (Figure 6). Beyond about twenty it is not very interesting because the maps all converge to Lambert's (1772) azimuthal equal area projection with a circular boundary.

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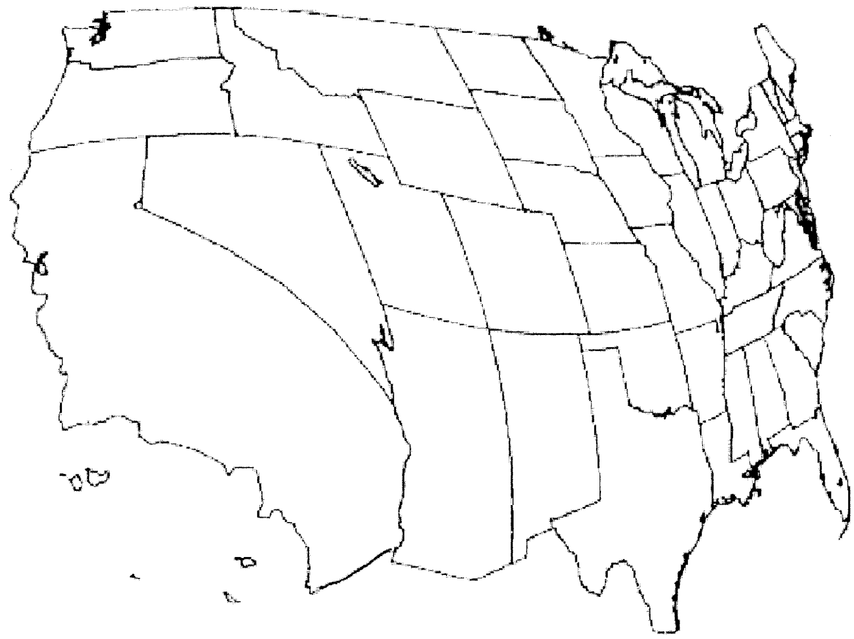

*Figure 5. The Santa Barbaran View. Cube root distance azimuthal projection centered on Santa Barbara.*

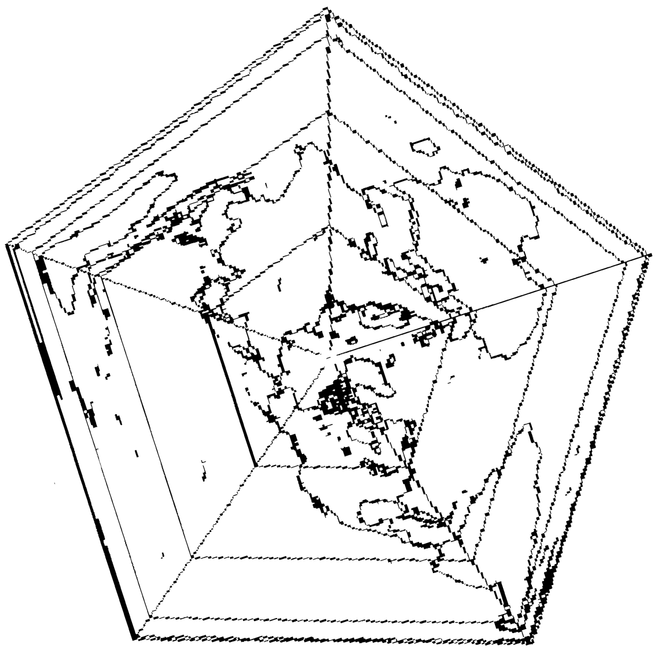

*Figure 6. A new equal area map in a polygon. This is the polar case in a pentagon.*

Maps on the five platonic solids have also been known for a long time (Fisher and Miller, 1944). They can be equal area or conformal. The gnomonic projection is particularly easy to do on the surface of these solids. Apparently they have never been done on the surface of a pyramid. The next illustration is a special case of an equal area projection having N pointed triangular protrusions on an N-sided base. For three lobes, the base is a triangle (this folds into a tetrahedron) with four lobes we get the pyramid (Figure 7). For six lobes the base is a hexagon. Again, all can be drawn using just one computer program with N as parameter. Conformal versions are also possible.

Composite equal area projections are perhaps of little value, but are fun. The combining technique works with most polycylindric and pseudocylindric projections including the Lambert cylindrical, Mollweide's (1805) projection and the sinusoidal, and those of Craster, Eckert, Boggs, etc., and with Tobler's (1974) hyperelliptical system of projections. All are equal area projections, all maintain the length of the equator, and all meridians meet the equator at a right angle. Therefore these projections can be joined at the equator to have one projection for the Northern hemisphere, and another for the Southern hemisphere. Figure 8 shows an example, with

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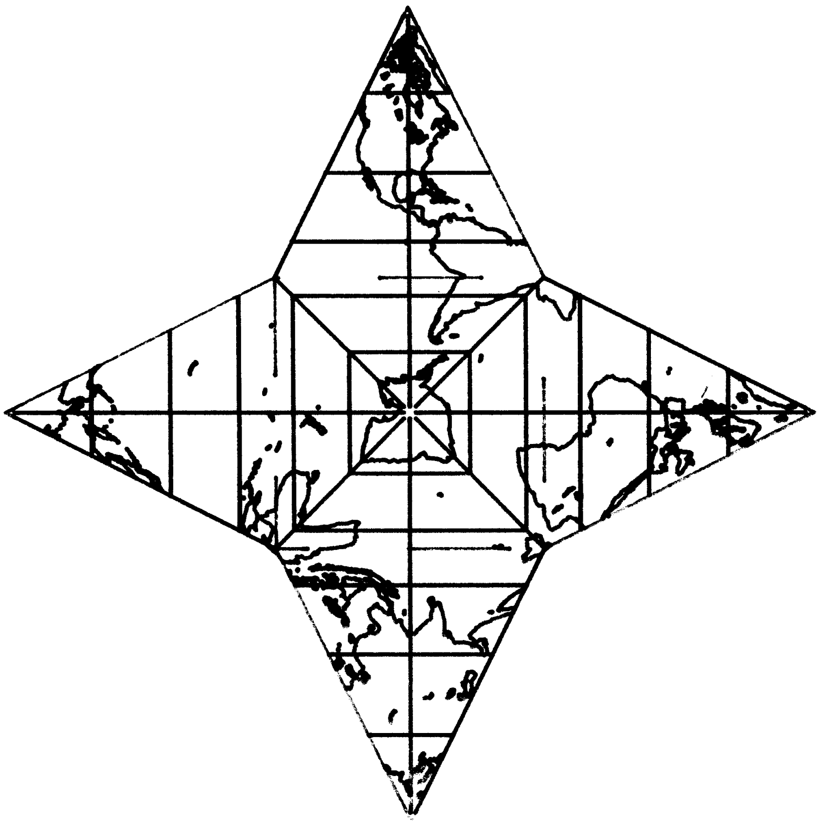

*Figure 7. An equal area projection on a pyramid (North polar case). (Cut out and glue together).*

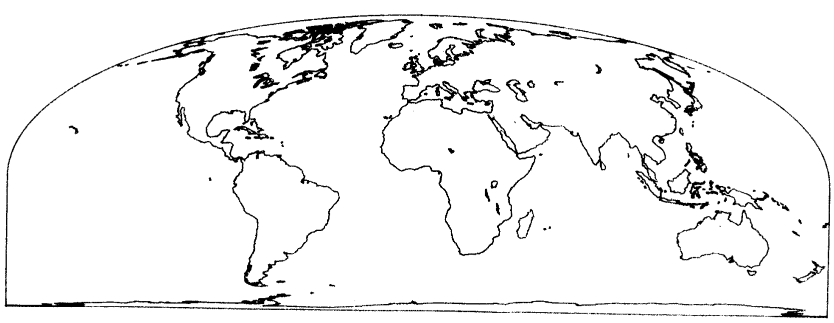

*Figure 8. An equal area projection combining two projections. Mollweide's projection above Lambert's cylindrical equal area projection.*

Mollweide's projection on top and Lambert's equal area cylindrical projection as the Southern base.

Affine transformations of equal area maps can yield more variants. An example is Mollweide's projection converted into an equal area circle (Figure 9). The equations are $X' = 2X$, $Y' = Y/2$, where X and Y are the original Mollweide coordinates and the primes denote the new coordinates. Another gives an equal area square obtained from Lambert's equal area cylindrical projection (Figure 10).

*"In addition to directions and areas, geographers who use maps are also concerned with distances."*

In addition to directions and areas, geographers who use maps are also concerned with distances. In general, all spherical distances cannot be correctly preserved on maps. But from one location, we have the equidistant azimuthal projection; the two-point equidistant projection is not often used but is occasionally appropriate. Chamberlin (1947) has given an approximate solution using three spherical distances. In order to best preserve all distances from more than three points one can used advanced techniques. Computing coordinates from distances is known as trilateration, it is also known as multi-dimensional scaling (Tobler, 1996). If one takes road distances from a Rand McNally (or other) road atlas and uses these distances to compute the location of the places, one can then interpolate the latitude-longitude graticule, and from this draw a map with state boundaries and coastlines. The resulting map projection (Figure 11) illustrates the distortion introduced by the road system.

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Furthermore, Tissot's (1881) indicatrix can be used to calculate the angular and areal distortion, as well as the distance distortion, in every direction, at each map location. These measures provide indications of the

*Figure 9. Mollweide's equal area projection affinely modified to fit in a circle. The equal area property is retained.*

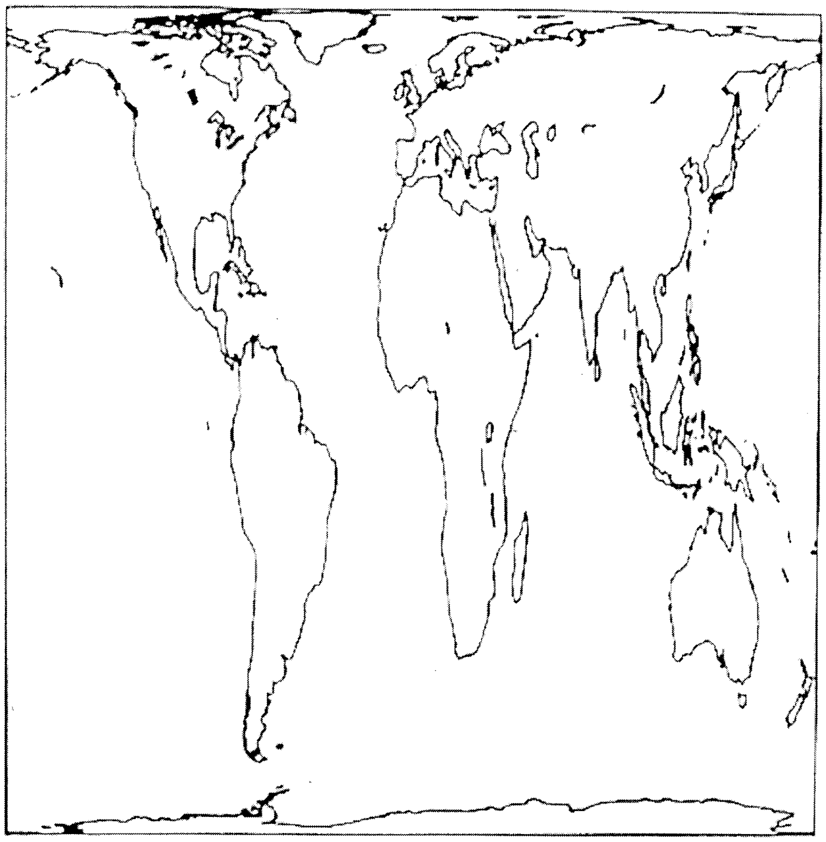

*Figure 10. Lambert's cylindrical equal area projection affinely modified to fit in a square. The equal area property is retained. The equations are $X' = X / \pi^{1/2}$, $Y' = \pi^{1/2} Y$, where X and Y are the original Lambert coordinates.*

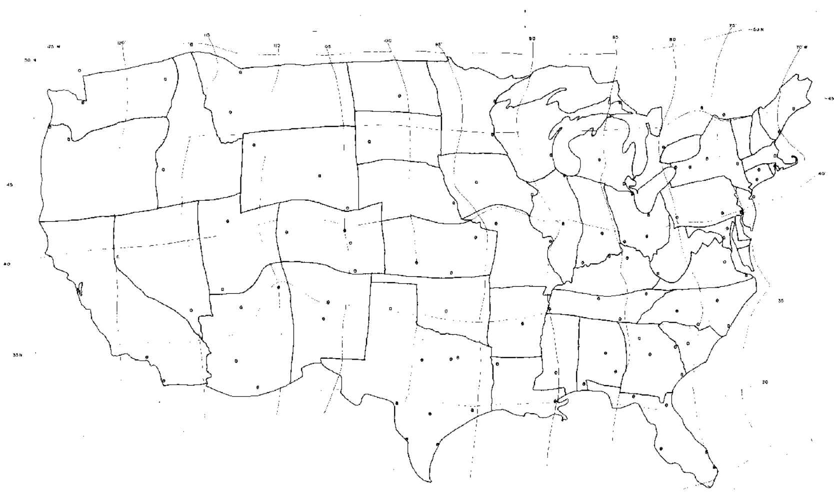

*Figure 11. Student rendition of a road distance map of the United States, fitting distances from an atlas table. Graticule and state boundaries interpolated.*

impacts of a road system, suggesting the use of map projections in transportation studies. Instead of using road distances, travel times or costs, or great circle distances, one can also construct a map to preserve, in the least squares sense, loxodromic (rhumb line) distances, a hypothesis being that Portolan Charts made prior to 1500 AD might have used such distances in their construction (Figure 12).

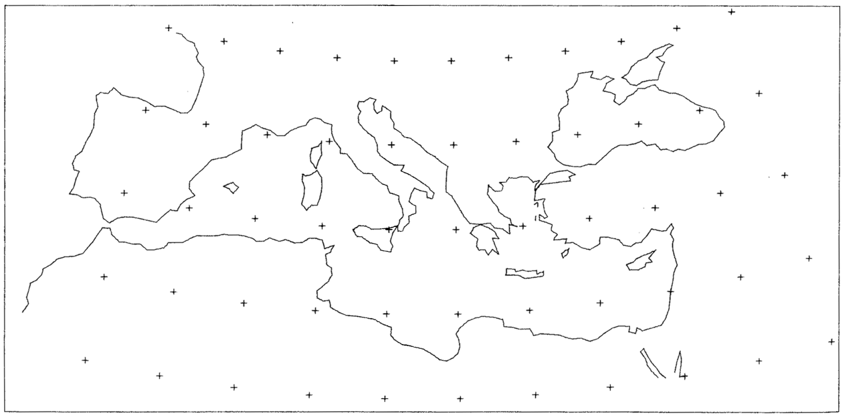

*Figure 12. Mediterranean Sea best preserving loxodromic distances.*

One additional projection that preserves distances is the Stab-Werner (1514) projection, but it shows distances correctly from only one central location. This is normally one of the poles, most often the North Pole. The projection also happens to be equal area. Oblique versions of Werner's projection are rare, although transverse versions of the closely related Bonne projection have been used. Such an oblique Werner projection is shown here (Figure 13) in the form of a graticule sketched in circa 1960 from line printer output with the center at the latitude and longitude of New York City, and with the central axis directed towards Seattle.

*"It is often asserted that transportation costs increase at a decreasing rate with geographic distance. In other words, that the cost-distance curve has a concave down shape."*

The North Pole can be seen, from the graticule, to north of the center of New York. The map has been rotated so that the New York – Seattle great circle is the horizontal axis. As such this is not a terribly interesting map but it suggests an alternative, as follows. It is often asserted that transportation costs increase at a decreasing rate with geographic distance. In other words, that the cost-distance curve has a concave down shape. On

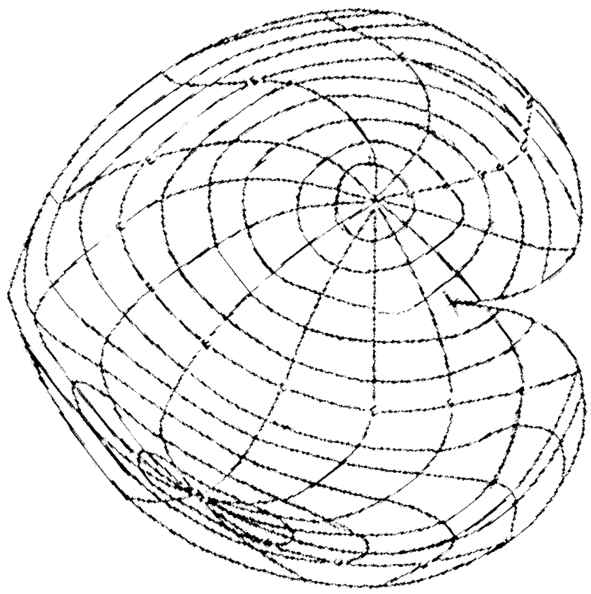

*Figure 13. Werner's projection centered at New York, with the central great circle directed towards the left tip. Seattle lies on this great circle at its correct distance from New York. The map is North oriented.*

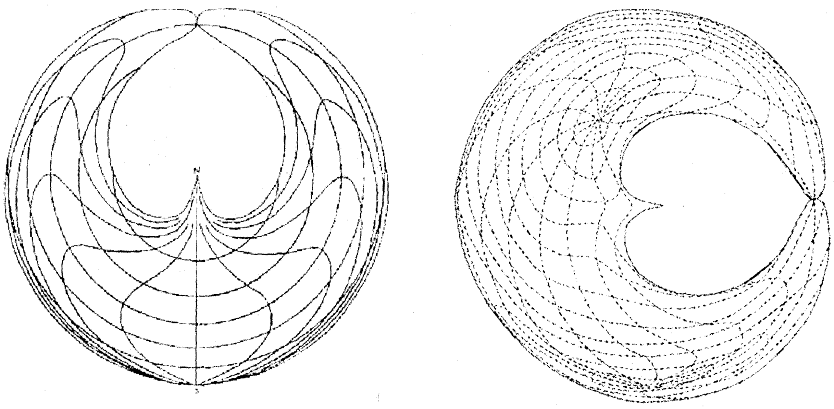

*Figure 14. An equal area map using concave down (square root of) spherical distances. Left: polar case graticule to illustrate the properties. Right: centered on New York with the central great circle directed towards Seattle. The map is North oriented.*

the map that follows (Figure 14) this cost idea is represented by the square root of the spherical distance from the map center, but the map has also been made to preserve spherical area.
The equations are:

$$X = R\,(2\rho)^{1/2}\sin(\lambda \sin\rho)$$
$$Y = R\,(2\rho)^{1/2}\cos(\lambda \sin\rho)$$

where $\rho$ is the spherical distance from the map center and $\lambda$ is the longitude. The map has a cordiform hole in the interior.  The latitude and longitude of New York has again been chosen as the center and the direction is to Seattle. It has again been rotated so that the New York – Seattle axis is horizontal. The equal area property, along with the concave distance function on this map, allows economic geography to be coupled with cartography. Other concave down distance functions can also be combined with the equal area condition to give difference maps of this type.

*"The equal area property, along with the concave distance function on this map, allows economic geography to be coupled with cartography."*

A common and useful technique is to use a correctly chosen coordinate system in order to simplify a problem. Instead of using straight meridians and parallels on a cylindrical map projection to show curved global satellite tracks, let us bend the meridians so that the satellite track becomes a straight line. This is more convenient for the automatic tracking of these satellites. What this looks like can be seen in an obscure paper by Breckman (1962) in which a map is designed for a satellite heading southeast from Cape Canaveral. The satellite path has become a straight line, making tracking much easier. Since the satellite does not cross over Antarctica this is therefore not on the map. The track is a 'saw-tooth' line, first South, then North, then South again.

On the next map (Figure 15) the geomagnetic coordinates are straightened in order to simplify the solution of problems involving terrestrial magnetism. This warps the normal geographic coordinates, but so what? It is not difficult to produce such maps graphically; it can also be done analytically. The idea is that we transform the graticule, and map, then study or solve our problem in this new reference frame, and then take the inverse transformation to bring the result back to the more conventional coordinates. This transform-solve-invert paradigm is well known in mathematics (Eves, 1980). This is also an example of how Mercator's idea works, and is one way in which areal cartograms, a generalization of equal area projections, may be used (Tobler, 2004). Kao (1967) provides further examples.

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For quickly displaying geographic data on a computer screen it is not necessary to use a complicated projection such as the transverse Merca-

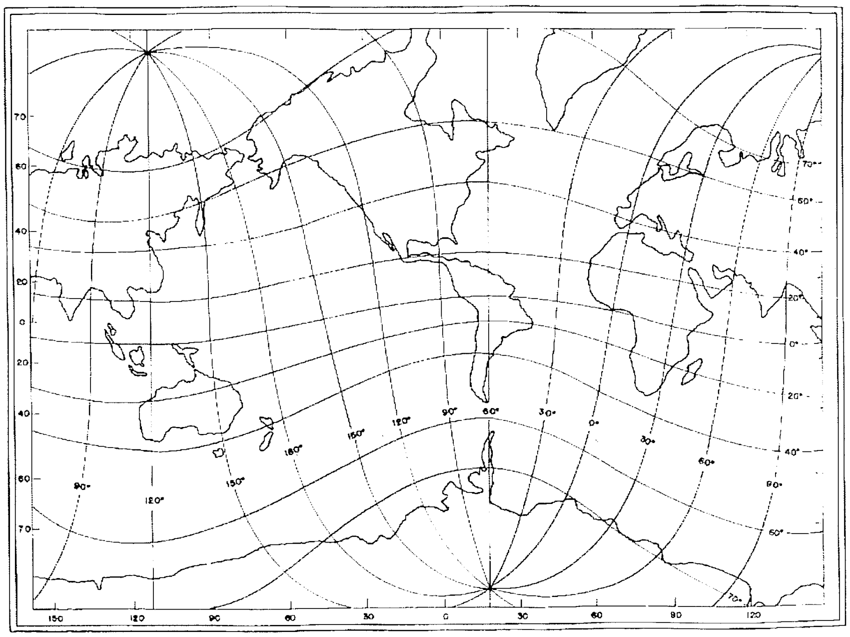

*Figure 15. Student rendition straightening magnetic meridians and parallels.*

tor. A much simpler set of equations will do, assuming that the data are in latitude and longitude coordinates (Tobler, 1974). Only two parameters are required: the average latitude and the average longitude of the center of the area. The necessary equations are then:

$X = R\{\cos(\varphi_o)\,\Delta\lambda - \sin(\varphi_o)\,\Delta\varphi\,\Delta\lambda\}$

$Y = R\{\Delta\varphi + 0.5\sin(\varphi_o)\cos(\varphi_o)\,\Delta\lambda\,\Delta\lambda\}$,

where R is in kilometers per degree on the mean radius sphere at the center location, $\Delta\varphi$ is the latitude minus the average latitude $\varphi_o$, and $\Delta\lambda$ is the longitude minus the average longitude. The X and Y values are then in kilometers. The resulting display is neither equal area nor conformal, but quite accurate and easy to compute for a small area not near either of the poles. The equatorial version for the entire earth – not a small area – will give a bow tie shaped map (Figure 16). Away from the Equator the whole earth can resemble a floppy bow tie. So use this projection only for areas smaller than the whole earth.

*"The resulting display is neither equal area nor conformal, but quite accurate and easy to compute for a small area not near either of the poles."*

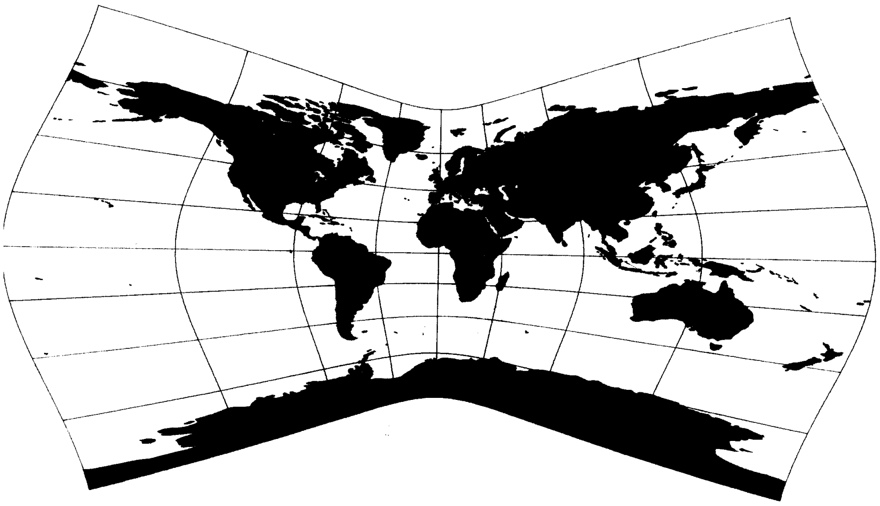

*Figure 16. A bow tie projection.*

It is sometimes asserted that one disadvantage of a globe is that the entire earth cannot be seen at one time. But, the entire earth can be seen at one time if the transformation $\varphi' = \varphi$ and $\lambda' = \lambda/2$ is used. Here $\varphi$ is latitude, and $\lambda$ is longitude. This transformation maps the entire surface of the earth onto one hemisphere. Repeat this for the backside of the globe and hardly anybody will notice that everything appears twice. East-West distances are of course foreshortened. Other versions of this are possible.

**Finally**

This introduction to a few unusual map projections will, hopefully, convince you that not only can these transformations be useful but also that they can be fun.

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